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# STABILITY OF SOLITARY WAVES FOR THE COUPLED BBM EQUATIONS

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## 1 Introduction

In this note, we consider large time behavior of the global solutions to the coupled BBM equations:

$$q_t - q_{txx} + r_x + (qr)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.1)$$

$$r_t - r_{txx} + q_x + qq_x + rr_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.2)$$

$$q(x, 0) = q_0(x), \quad r(x, 0) = r_0(x). \quad (1.3)$$

Here, for an integer  $s \geq 0$ ,  $H^s(\mathbb{R})$  denotes the space of functions  $u = u(x)$  such that  $\partial_x^l u$  are  $L^2$ -functions on  $\mathbb{R}$  for  $0 \leq l \leq s$ , endowed, with the norm  $\|\cdot\|_{H^s}$ , while  $H_a^1(\mathbb{R})$  is a space of functions whose element satisfy  $\|u\|_{H_a^1} \equiv \|e^{ax}u\|_{H^1} < \infty$  for  $a \in \mathbb{R}$ . We can write the system (1), (2) and (3) in the following system.

$$\partial_t u = Lu + f(u), \quad (1.4)$$

$$u(x, 0) = u_0(x) \quad (1.5)$$

where  $u = \begin{pmatrix} q \\ r \end{pmatrix}$  and,

$$L = \begin{pmatrix} 0 & -\partial_x(I - \partial_x^2)^{-1} \\ -\partial_x(I - \partial_x^2)^{-1} & 0 \end{pmatrix}, \quad f(u) = \begin{pmatrix} -\partial_x(I - \partial_x^2)^{-1}(qr) \\ -\frac{1}{2}\partial_x(I - \partial_x^2)^{-1}(q^2 + r^2) \end{pmatrix}.$$

The BBM equations have two parameter family of solitary wave. In [1], they show that solitary waves  $u_{c_0}(x - c_0 t + \gamma_0) = \phi_{c_0}(x - c_0 t + \gamma_0)(1, 1)^T$  exist for any speed  $c_0 > 1$  and shift  $\gamma_0 \in \mathbb{R}$ . Explicitly,

$$\begin{aligned} \phi_{c_0}(x) &= \frac{3(c_0 - 1)}{2} \operatorname{sech}^2\left(\frac{1}{2}\nu x\right) \\ &= O(\exp(-\nu|x|)) \quad (|x| \rightarrow \pm\infty), \end{aligned} \quad (1.6)$$

where  $\nu = \sqrt{\frac{c_0 - 1}{c_0}}$ . By using the Lyapunov theory, they derive the following theorem on orbital stability.

**Theorem 1.1.** *Let  $c_0 > 1$ . For any  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $u \in C([0, t_0]; H^1(\mathbb{R}))$  is a solution to (1.4) and (1.5) with  $\|u_0 - u_{c_0}\|_{H^1} \leq \delta$ , then  $u(t)$  can be extended to a solution in  $0 \leq t < +\infty$ , and*

$$\sup_{t \geq 0} \inf_{\xi \in \mathbb{R}} \|u(\cdot, t) - u_{c_0}(\cdot - \xi)\|_{H^1} \leq \epsilon. \quad (1.7)$$

However we can't expect that if the initial data  $u_0(x)$  is close to some solitary wave  $u_{c_0}(x + \gamma_0)$  with speed  $c_0$ , then the solution tends to the translate of same solitary wave as  $t$  goes to  $\infty$  asymptotically. Our aim is to describe the long-time asymptotic behavior of solutions initially close to a solitary wave. Main result is following. Convergence in  $H_a^1$  means local uniform convergence.

**Theorem 1.2.** *Let  $0 < a < \nu$ . We assume  $u_0(x) = u_{c_0}(x + \gamma_0) + v_0(x) \in H^2 \cap H_a^1$ . There exist  $\epsilon > 0$ ,  $c_1 > 1$  and  $b > 0$  such that if  $c_0 \in (1, c_1)$  and  $\|v_0\|_{H^1} + \|v_0\|_{H_a^1} < \epsilon$ , then*

$$\|u(\cdot, t) - u_{c_+}(\cdot - c_+t + \gamma_+)\|_{H^1} \leq C\epsilon, \quad (1.8)$$

$$\|u(\cdot + c_+t - \gamma_+, t) - u_{c_+}(\cdot)\|_{H_a^1} \leq C\epsilon e^{-bt}, \quad (1.9)$$

for some  $c_+ > 1$ ,  $\gamma_+ \in \mathbb{R}$  with  $|c_0 - c_+| < C\epsilon$ ,  $|\gamma_0 - \gamma_+| < C\epsilon$ .

We remark that the estimate similar to (6) and (7) were obtained for other types of equation (see [2], [3], [4] and [5]).

## 2 Spectrum

In order to prove Theorem 1.2, we derive the equation for the perturbation. We put  $y = x - \int_0^t c(s)ds + \gamma(t)$  and  $u(x, t) = u_{c(t)}(y) + v(y, t)$ . Then  $v(y, t)$  satisfies

$$\partial_t v = Av + F, \quad (2.1)$$

where  $v = \begin{pmatrix} \rho \\ \eta \end{pmatrix}$  and,

$$\begin{aligned} A &= \partial_y(I - \partial_y^2)^{-1}L_{c_0}, \\ L_{c_0} &= \begin{pmatrix} c_0(I - \partial_y^2) - \phi_{c_0} & -(1 + \phi_{c_0}) \\ -(1 + \phi_{c_0}) & c_0(I - \partial_y^2) - \phi_{c_0} \end{pmatrix}, \\ F &= -(\dot{\gamma}\partial_y\phi_c + \dot{c}\partial_c\phi_c) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c - c_0 - \dot{\gamma})\partial_y v \\ &\quad - \partial_y(I - \partial_y^2)^{-1}(\phi_c - \phi_{c_0}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v + \begin{pmatrix} -\partial_y(I - \partial_y^2)^{-1}\rho\eta \\ -\frac{1}{2}\partial_y(I - \partial_y^2)^{-1}(\rho^2 + \eta^2) \end{pmatrix}. \end{aligned} \quad (2.2)$$

To study the weighted perturbation  $e^{ay}v$ , we deal with the spectrum of the operator  $A_a = e^{ay}Ae^{-ay}$ . We can see that if  $0 < a < \nu$ , then the essential spectrum of  $A_a$  lie in the open left half plane. We put

$$A_a^\infty = D_a(I - D_a^2)^{-1} \begin{pmatrix} c_0(I - D_a^2) & -1 \\ -1 & c_0(I - D_a^2) \end{pmatrix}. \quad (2.3)$$

Since  $\phi_{c_0}$  decays exponentially as  $|y| \rightarrow \infty$ , we obtain from (2.2) and (2.3)

$$\sigma_{ess}(A_a) = \sigma_{ess}(A_a^\infty). \quad (2.4)$$

Applying the Fourier transform to  $A_a^\infty$ , we see

$$\sigma_{ess}(A_a^\infty) = \left\{ z \in \mathbb{C} \mid z = \frac{(ik - a)(-c_0(ik - a)^2 + c_0 \pm 1)}{1 - (ik - a)^2}, k \in \mathbb{R} \right\}. \quad (2.5)$$

Therefore, it follows that

$$\operatorname{Re} \sigma_{ess}(A_a) \leq -ac + \frac{1}{2(1 + a^2)} := -b_* < 0. \quad (2.6)$$

Hence, if  $0 < a < \nu$ , then the essential spectrum of  $A_a$  lie in the open left half plane.

**Proposition 2.1.** *Let  $0 < a < \nu$ .*

- (1) *There exists  $\nu_* \in (0, 1)$  such that for all  $\nu \in (0, \nu_*)$ , the only eigenvalue  $\lambda$  of  $A$  with  $\operatorname{Re} \lambda \geq 0$  is  $\lambda = 0$  and  $\lambda = 0$  is the eigenvalue with algebraic multiplicity 2.*
- (2) *For each  $\nu \in (0, \nu_*)$ , there exists  $\epsilon(\nu) > 0$  such that the only eigenvalue  $\lambda$  of  $A_a$  with  $\operatorname{Re} \lambda \geq -\epsilon(\nu)$  is  $\lambda = 0$ .*

We put

$$-b_{\max} = \inf \{ -b \mid \lambda = 0 \text{ is the only eigenvalue of } A_a \text{ with } \operatorname{Re} \lambda \geq -b > -b_* \}.$$

Then, the decay estimates for the linearized equations (2.7) below play a crucial role in our analysis. By using (2.6) and The Prüss's theorem [6], we obtain the following.

**Proposition 2.2.** *Assume  $0 < a < \nu$  and  $\lambda = 0$  is the only eigenvalue of  $A$  in the closed right half plane. Then the problem*

$$\begin{cases} w_t = A_a w \\ w(x, 0) = w_0(x) \in \operatorname{range} Q \end{cases} \quad (2.7)$$

*has a solution with*

$$\|w(\cdot, t)\|_{H_a^1} \leq C e^{-bt} \|w_0\|_{H_a^1} \quad (2.8)$$

*for some  $b$  with  $0 < b < b_{\max}$ . Here  $Q$  is a projection onto  $\operatorname{Ker}\{A_a^*\}^\perp$  and  $A_a^*$  is the adjoint operator of  $A_a$ .*

Next, we study a basis for the generalized zero eigen space  $\text{Ker}_g(A_a)$ . We can see

$$\text{Ker}_g(A_a) = \text{span}\{\partial_y u_{c_0}, \partial_c u_{c_0}\}. \quad (2.9)$$

The solitary wave  $u_{c_0}$  satisfies the following.

$$\begin{pmatrix} c_0(I - \partial_y^2) + \frac{1}{2}\phi_{c_0} & -1 + \frac{1}{2}\phi_{c_0} \\ -1 + \frac{1}{2}\phi_{c_0} & c_0(I - \partial_y^2) + \frac{1}{2}\phi_{c_0} \end{pmatrix} u_{c_0} = 0. \quad (2.10)$$

Differentiating (2.10) with  $y$  and  $c$ , we obtain

$$L_{c_0} \partial_y u_{c_0} = 0, \quad (2.11)$$

and

$$L_{c_0} \partial_c u_{c_0} = -(1 - \partial_y^2) u_{c_0}. \quad (2.12)$$

From (2.11), we get

$$A \partial_y u_{c_0} = 0. \quad (2.13)$$

Hence, we obtain  $\partial_y u_{c_0} \in \text{Ker}_g(A)$ . From (2.12), we have

$$A \partial_c u_{c_0} = -\partial_y u_{c_0}. \quad (2.14)$$

It follows that

$$A^2 \partial_c u_{c_0} = 0. \quad (2.15)$$

Hence, we obtain (2.9). Let us introduce  $\tilde{\xi}_1, \tilde{\xi}_2$  by

$$\tilde{\xi}_1 = \partial_y u_{c_0}, \quad \tilde{\xi}_2 = \partial_c u_{c_0}.$$

We take biorthogonal bases  $\{\tilde{\xi}_1, \tilde{\xi}_2\}$  and  $\{\tilde{\eta}_1, \tilde{\eta}_2\}$  for  $\text{Ker}_g(A)$  and  $\text{Ker}_g(A^*)$ .

Let

$$\xi_i = e^{ay} \tilde{\xi}_i, \quad \eta_i = e^{-ay} \tilde{\eta}_i,$$

for  $i = 1, 2$ . Then  $\{\xi_1, \xi_2\}$  and  $\{\eta_1, \eta_2\}$  are biorthogonal bases for  $\text{Ker}_g(A_a)$  and  $\text{Ker}_g(A_a^*)$ .

### 3 Modulation Equation

To obtain the decay estimate of perturbation  $w = e^{ay} v$ , we need to let  $w$  belong to orthogonal to  $\text{Ker}_g(A_a^*)$ . The following Proposition 3.1 concerned the existence of a decomposition  $u(x, t) \mapsto (v(y, t), \gamma(t), c(t))$  satisfies the condition (3.3) below.

**Proposition 3.1.** *Let  $0 < a < \nu$  and  $T \geq 0$ . Then there exists  $\delta_0, \delta_1 > 0$  such that for any  $\gamma_0 \in \mathbb{R}$ , if  $e^{ax}u(x) \in C([0, T], H^1)$  and*

$$\sup_{0 \leq t \leq T} \|e^{a(\cdot + \gamma_0)}(u(\cdot, t) - u_{c_0}(\cdot - c_0 t + \gamma_0))\|_{H^1} < \delta_0 \quad (3.1)$$

*holds, then there exists a unique function  $t \rightarrow (\gamma(t), c(t)) \in C([0, t_1], \mathbb{R}^2)$  with*

$$\sup_{0 \leq t \leq t_1} |\gamma(t) - \gamma_0| + |c(t) - c_0| < \delta_1 \quad (3.2)$$

*such that*

$$\int_{\mathbb{R}} [u(x, t) - u_{c(t)}(y)] e^{ay} \eta_k(y) dy = 0, \quad (3.3)$$

*for  $k = 1, 2$  and  $0 \leq t \leq T$ .*

**Proposition 3.2.** *There exist  $\delta_2, \delta_3 > 0$  such that for any  $T > 0$ , if  $e^{ax}u(x) \in C([0, T], H^1)$  and*

$$\sup_{0 \leq t \leq T} \|e^{ay}v(y, t)\|_{H^1} \leq \delta_2, \quad (3.4)$$

*and*

$$\sup_{0 \leq t \leq T} |c(t) - c_0| \leq \delta_3, \quad (3.5)$$

*hold, then a unique extension of  $(\gamma, c) \in C([0, T + t_*], \mathbb{R}^2)$  exists for some  $t_* > 0$  with*

$$\int_{\mathbb{R}} [u(x, t) - u_{c(t)}(y)] e^{ay} \eta_k(y) dy = 0, \quad (3.6)$$

*for  $k = 1, 2$  and  $0 \leq t \leq T + t_*$ .*

To estimate the weighted perturbation, we need to estimate  $\dot{\gamma}$  and  $\dot{c}$ . We shall derive the modulation equations (3.11) below. We put  $\tau = \int_0^t c(s) ds - \gamma(t)$ . Then,  $w$  satisfies

$$w_\tau = \frac{1}{c_0} A_a w + J \quad (3.7)$$

where

$$J = -\frac{1}{c - \dot{\gamma}} e^{ay} (\dot{\gamma} \partial_y u_c + \dot{c} \partial_c u_c) + \tilde{J}, \quad (3.8)$$

where  $\tilde{J} = J_1 + J_2 + J_3$ ,  $D_a = \partial_y - a$  and

$$\begin{aligned} J_1 &= \frac{1}{c_0} D_a (I - D_a^2)^{-1} (\phi_{c_0} - \phi_c) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} w, \\ J_2 &= \frac{1}{c_0} \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} D_a (I - D_a^2)^{-1} \left\{ \phi_c \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} w, \\ J_3 &= -\frac{1}{c - \dot{\gamma}} D_a (I - D_a^2)^{-1} \begin{pmatrix} e^{ay} q r \\ -\frac{1}{2} e^{ay} (q^2 + r^2) \end{pmatrix}. \end{aligned} \quad (3.9)$$

$P$  denotes the projection onto the zero eigenspace of  $A_a$ . In order to prove the decay estimate (1.9), we require  $PJ = 0$ . Then, we obtain

$$\langle \eta_i, J \rangle = 0. \quad (3.10)$$

for  $i = 1, 2$ , where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$  inner product. It follows from (3.8) that

$$-\frac{1}{c - \dot{\gamma}} \{ \langle \eta_i, e^{ay} \partial_y u_c \rangle \dot{\gamma} + \langle \eta_i, e^{ay} \partial_c u_c \rangle \dot{c} \} + \langle \eta_i, \tilde{J} \rangle = 0.$$

We put  $e_1 = \partial_y u_c - \partial_y u_{c_0}$ ,  $e_2 = \partial_c u_c - \partial_c u_{c_0}$ . Since  $\langle \tilde{\eta}_i, \tilde{\xi}_j \rangle = \delta_{ij}$ , we derive

$$\begin{pmatrix} 1 + \langle \tilde{\eta}_1, e_1 \rangle & \langle \tilde{\eta}_1, e_2 \rangle \\ \langle \tilde{\eta}_2, e_1 \rangle & 1 + \langle \tilde{\eta}_2, e_2 \rangle \end{pmatrix} \begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = c_0 \left( 1 - \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right)^{-1} \begin{pmatrix} \langle \eta_1, \tilde{J} \rangle \\ \langle \eta_2, \tilde{J} \rangle \end{pmatrix}.$$

Hence, we obtain

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(|c(t) - c_0|) \right\} \begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = c_0 \left( 1 - \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right)^{-1} \begin{pmatrix} \langle \eta_1, \tilde{J} \rangle \\ \langle \eta_2, \tilde{J} \rangle \end{pmatrix}.$$

Therefore, if  $|c(t) - c_0| + |\dot{\gamma}(t)|$  is small, then it follows that

$$\begin{aligned} |\dot{\gamma}(t)| + |\dot{c}(t)| &\leq C \|\tilde{J}\|_{L^2} \\ &\leq C(\|\tilde{J}_1\|_{L^2} + \|\tilde{J}_2\|_{L^2} + \|\tilde{J}_3\|_{L^2}). \end{aligned} \quad (3.11)$$

## 4 Energy estimate

In order to prove the decay estimate given by Proposition 5.1, we prepare the following Proposition 4.1. The proposition is concerned with the energy estimate for the BBM equations (1.1), (1.2) and (1.3). A solitary wave  $u_{c_0}$  is a critical point of energy functional

$$E[u; c] = H[u] - c_0 N[u] \quad (4.1)$$

where  $u = (q, r)^T$  and

$$H[u] = \int_{\mathbb{R}} \left( qr + \frac{1}{2} q^2 r + \frac{1}{6} r^3 \right) dx, \quad (4.2)$$

$$N[u] = \frac{1}{2} \int_{\mathbb{R}} (q^2 + r^2 + q_x^2 + r_x^2) dx. \quad (4.3)$$

We denote  $H[u]$  and  $N[u]$  are conserved integrals of BBM equations.

**Proposition 4.1.** *If  $|c(t) - c_0| + \|v(\cdot, t)\|_{H^1}$  is sufficiently small for  $0 \leq t \leq T$ , then we have*

$$\|v(\cdot, t)\|_{H^1} \leq C(\sqrt{\delta E} + |c(t) - c_0| + \|w(\cdot, t)\|_{L^2}) \quad (4.4)$$

for  $0 \leq t \leq T$ , where  $\delta E = E[u(\cdot, 0)] - E[u_{c_0}(\cdot)]$ .

PROOF. Put  $z(y, t) = (z_1, z_2)^T = u(x, t) - u_{c_0}(y)$ . Since  $\delta E$  is constant in time, it follows that

$$\begin{aligned}\delta E &= E[z(y, t) + u_{c_0}(y)] - E[u_{c_0}(y)] \\ &= -\frac{1}{2} \int_{\mathbb{R}} z^T L_{c_0} z dy + \frac{1}{2} \int_{\mathbb{R}} z_1^2 z_2 dy + \frac{1}{6} \int_{\mathbb{R}} z_2^3 dy.\end{aligned}\quad (4.5)$$

From the Sobolev inequality, we have

$$\begin{aligned}|\frac{1}{2} \int_{\mathbb{R}} z_1^2 z_2 dy + \frac{1}{6} \int_{\mathbb{R}} z_2^3 dy| &\leq C \|z\|_{H^1}^3 \\ &\leq C(|c(t) - c_0| + \|v\|_{H^1})^3,\end{aligned}\quad (4.6)$$

where  $z(y, t) = v(y, t) + u_{c(t)}(y) - u_{c_0}(y)$ .

We obtain

$$\begin{aligned}-\frac{1}{2} \int_{\mathbb{R}} z^T L_{c_0} z dy &= -\frac{1}{2} \int_{\mathbb{R}} v^T L_{c_0} v dy - \int_{\mathbb{R}} v^T L_{c_0} (u_c(y) - u_{c_0}(y)) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} (u_c(y) - u_{c_0}(y))^T L_{c_0} (u_c(y) - u_{c_0}(y)) dy \\ &\leq -\frac{1}{2} \int_{\mathbb{R}} v^T L_{c_0} v dy + \left| \int_{\mathbb{R}} e^{-ay} w^T L_{c_0} (u_c(y) - u_{c_0}(y)) dy \right| \\ &\quad + \frac{1}{2} \left| \int_{\mathbb{R}} (u_c(y) - u_{c_0}(y))^T L_{c_0} (u_c(y) - u_{c_0}(y)) dy \right| \\ &\leq -\frac{1}{2} \int_{\mathbb{R}} v^T L_{c_0} v dy + C(|c(t) - c_0|^2 + \|w\|_{L^2}^2),\end{aligned}\quad (4.7)$$

and

$$\begin{aligned}-\frac{1}{2} \int_{\mathbb{R}} v^T L_{c_0} v dy &= -\frac{1}{2} c_0 (\|v(\cdot, t)\|_{L^2} + \|v_y(\cdot, t)\|_{L^2}) - \int_{\mathbb{R}} (1 + \phi_{c_0}) v_1 v_2 dy \\ &\quad - \int_{\mathbb{R}} \phi_{c_0} v^T v dy \\ &\leq -\frac{1}{2} c_0 (\|v(\cdot, t)\|_{L^2} + \|v_y(\cdot, t)\|_{L^2}) + \frac{1}{2} \|v(\cdot, t)\|_{L^2}^2 \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-ay} \phi_{c_0} w^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v dy \\ &\leq -\frac{1}{2} \{(c_0 - 1) \|v(\cdot, t)\|_{L^2}^2 + c_0 \|v_y(\cdot, t)\|_{L^2}^2\} + C \|v\|_{L^2} \|w\|_{L^2} \\ &\leq -\frac{1}{2} \{\frac{1}{2} (c_0 - 1) \|v(\cdot, t)\|_{L^2}^2 + c_0 \|v_y(\cdot, t)\|_{L^2}^2\} + C \|w(\cdot, t)\|_{L^2}^2.\end{aligned}\quad (4.8)$$

Summarizing (4.5), (4.6), (4.7) and (4.8), we get

$$\frac{1}{2} \{(c_0 - 1) \|v(\cdot, t)\|_{L^2}^2 + c_0 \|v_y(\cdot, t)\|_{L^2}^2\} \leq C(|\delta E| + |c(t) - c_0|^2 + \|w(\cdot, t)\|_{L^2}^2). \quad (4.9)$$

This completes the proof.  $\square$



## 5 Decay estimate

**Proposition 5.1.** *There exist  $\delta_4 > 0$  and  $\epsilon_* > 0$  such that if the decomposition exists for  $t \in [0, T]$  and the following conditions hold:*

$$(i) \quad \sqrt{|\delta E|} + \|w(\cdot, t)\|_{H^1} + |c(t) - c_0| + \left| \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right| + \|v(\cdot, t)\|_{H^1} \leq \delta_4, \quad (5.1)$$

$$(ii) \quad |c(0) - c_0| + \sqrt{|\delta E|} + \|w(\cdot, 0)\|_{H^1} \leq \epsilon \leq \epsilon_*, \quad (5.2)$$

then we have

$$e^{\kappa bt} \|w(\cdot, t)\|_{H^1} + |c(t) - c_0| + \left| \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right| + \|v(\cdot, t)\|_{H^1} \leq C\epsilon, \quad (5.3)$$

where  $\kappa = 1 + 2\delta_4/c_0$ .

**PROOF.** First, we evaluate  $|\dot{\gamma}(t)| + |\dot{c}(t)|$ . If  $|c(t) - c_0| + \left| \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right|$  is small, then we have from (3.11) and (3.9)

$$\begin{aligned} |\dot{\gamma}(t)| + |\dot{c}(t)| &\leq C(\|J_1\|_{L^2} + \|J_2\|_{L^2} + \|J_3\|_{L^2}) \\ &\leq C(|c(t) - c_0| + |\dot{\gamma}(t)| + \|v(\cdot, t)\|_{H^1}) \|w(\cdot, t)\|_{H^1}. \end{aligned} \quad (5.4)$$

Therefore, We get from (5.4) and (5.1)

$$|\dot{\gamma}(t)| + |\dot{c}(t)| \leq C\delta_4^2. \quad (5.5)$$

Next, we evaluate  $\|w(\cdot, t)\|_{H^1}$ . By the Duhamel principle, we see from (3.7)

$$w(\tau) = e^{\frac{1}{c_0} A_a \tau} w(0) + \int_0^\tau e^{\frac{1}{c_0} A_a (\tau-s)} QJ(s) ds. \quad (5.6)$$

Let  $b < b' < b_{max}$ . For  $0 \leq \tau \leq \tau(T)$ , we find from Proposition 2.2

$$\|w(\cdot, \tau)\|_{H^1} \leq C e^{-\frac{b'}{c_0} \tau} \|w(\cdot, 0)\|_{H^1} + \int_0^\tau e^{-\frac{b'}{c_0} (\tau-s)} \|QJ\|_{H^1} ds. \quad (5.7)$$

We obtain from (3.8), (3.9) and (5.4)

$$\begin{aligned} \|J\|_{H^1} &\leq C \left( 1 + \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right) (|\dot{\gamma}(t)| + |\dot{c}(t)|) + C\delta_4 \|w\|_{H^1} \\ &\leq C\delta_4 \|w\|_{H^1}. \end{aligned} \quad (5.8)$$

Now, we define  $M(T)$  by

$$M(T) = \sup_{0 \leq \tau \leq \tau(T)} e^{\frac{b}{c_0} \tau} \|w(\cdot, \tau)\|_{H^1}.$$

It follows from (5.7) and (5.8) that

$$\begin{aligned} e^{\frac{b}{c_0}\tau} \|w(\cdot, \tau)\|_{H^1} &\leq C \|w(\cdot, 0)\|_{H^1} + C\delta_4 M(T) \int_0^\tau e^{-\frac{b'-b}{c_0}(\tau-s)} ds \\ &\leq C \|w(\cdot, 0)\|_{H^1} + C\delta_4 M(T). \end{aligned} \quad (5.9)$$

Therefore, if  $\delta_4$  is small, then (5.9) gives the desired estimate

$$M(T) \leq C \|w(\cdot, 0)\|_{H^1}. \quad (5.10)$$

In order to evaluate  $|c(\tau) - c_0|$ , we shall use

$$\begin{aligned} c(\tau) - c(0) &= \int_0^\tau \frac{dc}{ds}(s) ds \\ &= \int_0^\tau \dot{c}(s) \frac{dt}{ds}(s) ds. \end{aligned} \quad (5.11)$$

By using (5.2) and (5.4), we have

$$\begin{aligned} |c(\tau) - c_0| &\leq |c(0) - c_0| + \sup_{0 \leq \tau \leq \tau(T)} \left| \frac{1}{c - \dot{\gamma}} \right| \int_0^\tau |\dot{c}(s)| d\tau \\ &\leq |c(0) - c_0| + C\delta_4 M(T) \int_0^\tau e^{-bs} ds \\ &\leq C(|c(0) - c_0| + \|w(\cdot, 0)\|_{H^1}) \\ &\leq C\epsilon. \end{aligned} \quad (5.12)$$

Next, we consider  $\|v(\cdot, t)\|_{H^1}$ . From Proposition 4.1, we get

$$\begin{aligned} \|v(\cdot, t)\|_{H^1} &\leq C(\sqrt{|\delta E|} + |c(t) - c_0| + \|w(\cdot, t)\|_{H^1}) \\ &\leq C\epsilon. \end{aligned} \quad (5.13)$$

Finally, we deal with  $\left| \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right|$ . From (5.12) and (5.4), it follows that

$$\begin{aligned} \left| \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right| &\leq \frac{|c - c_0| + |\dot{\gamma}|}{|c| - |\dot{\gamma}|} \\ &\leq C\epsilon + C\|w(\cdot, t)\|_{H^1} \\ &\leq C\epsilon. \end{aligned} \quad (5.14)$$

Summarizing (5.10), (5.12), (5.13) and (5.14), we obtain (5.3). This completes the proof.  $\square$

## 6 Proof of Theorem 1.2

PROOF. We assume that

$$u_0(x) = u(x, 0) = u_{c_0}(x + \gamma_0) + v_0(x) \in H^2 \cap H_a^1. \quad (6.1)$$

Since the map  $t \mapsto \|u(\cdot, t)\|_{H^1} + \|u(\cdot, t)\|_{H_a^1}$  is continuous for  $t \in [0, \infty)$ , there exist  $\epsilon_1 > 0$  and  $t_1 > 0$  such that if  $\|u(\cdot, 0) - u_{c_0}(\cdot + \gamma_0)\|_{H_a^1} \leq \epsilon_1$ , then

$$\sup_{0 \leq t \leq t_1} \|e^{a(\cdot + \gamma_0)}(u(\cdot, t) - u_{c_0}(\cdot - c_0 t + \gamma_0))\|_{H^1} \leq \delta_0, \quad (6.2)$$

where  $\delta_0$  is as in Proposition 3.1. Hence, there exists the decomposition  $u(x, t) \mapsto (v(y, t), \gamma(t), c(t))$  exists for  $0 \leq t \leq t_1$ .

Next, we shall prove (5.2). If  $\|e^{ax}v_0\|_{H^1}$  is small, then the map  $u(\cdot, 0) \mapsto (\gamma(0), c(0))$  is locally Lipschitz on  $H_a^1$  and  $u_{c_0}(\cdot + \gamma_0) \mapsto (\gamma_0, c_0)$ . Therefore, we obtain

$$\begin{aligned} |\gamma(0) - \gamma_0| + |c(0) - c_0| &\leq C\|u(\cdot, 0) - u_{c_0}(\cdot + \gamma_0)\|_{H_a^1} \\ &= C\|v_0\|_{H_a^1}. \end{aligned} \quad (6.3)$$

Also it follows from (6.3) that

$$\begin{aligned} \|w(\cdot, 0)\|_{H^1} &= \|u(\cdot, t) - u_{c(0)}(\cdot + \gamma(0))\|_{H_a^1} \\ &\leq \|u(\cdot, t) - u_{c_0}(\cdot + \gamma_0)\|_{H_a^1} + \|u_{c_0}(\cdot + \gamma_0) - u_{c(0)}(\cdot + \gamma(0))\|_{H_a^1} \\ &\leq C\|v_0\|_{H_a^1}, \end{aligned} \quad (6.4)$$

and we get from (4.5)

$$|\delta E| \leq C\|v_0\|_{H_1}^2. \quad (6.5)$$

Hence, if  $\|v_0\|_{H^1} + \|v_0\|_{H_a^1}$  is small, then (5.2) holds.

Next, we derive (5.1). From (5.14), (5.12) and (5.4), we obtain

$$\begin{aligned} \left| \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}} \right| &\leq C(\sqrt{\delta E} + |c(0) - c_0| + \|w(\cdot, 0)\|_{H^1}) \\ &\leq C\|v_0\|_{H_a^1}. \end{aligned} \quad (6.6)$$

Since the left-hand side of (5.1) is continuous in  $t$ , there exist  $\epsilon_2 > 0$  and  $t_2 > 0$  ( $t_2 < t_1$ ) such that if  $\|v_0\|_{H^1} + \|v_0\|_{H_a^1} < \epsilon_2$ , then (5.1) holds for  $0 \leq t \leq t_2$ .

We put

$$T_{\max} = \sup\{T > 0 \mid \text{the decomposition (3.3) and (5.1) hold for } 0 \leq t \leq T\}.$$

If  $T_{\max} = +\infty$ , then we can show Theorem 1.2. If  $T_{\max} < +\infty$ , then we let  $C\epsilon < \min\{\delta_2, \delta_3, \delta_4/2\}$  where  $C$  is as in Proposition 5.1. From proposition 5.1, if  $|c(0) - c_0| + \sqrt{|\delta E|} + \|w(\cdot, t)\|_{H^1} < \epsilon$ , then we have

$$\begin{aligned}\|w(\cdot, t)\|_{H^1} &< C\epsilon < \delta_2, \\ |c(t) - c_0| &< C\epsilon < \delta_3,\end{aligned}$$

for  $0 \leq t \leq T$ . From Proposition 3.2, the decomposition can be extended. Then, it follows that there exists  $t_3$  such that the decomposition and the estimate (5.1) hold for  $0 \leq t \leq T_{\max} + t_3$ . This contradicts the definition of  $T_{\max}$ . Hence  $T_{\max} = +\infty$ .

Finally, we shall prove (1.9). From (5.4), we have

$$\begin{aligned}|\dot{c}| + |\dot{\gamma}| &\leq C\|w(\cdot, t)\|_{H^1} \\ &\leq C\epsilon e^{-bt}.\end{aligned}\tag{6.7}$$

Therefore, there exists  $c_+ = \lim_{t \rightarrow \infty} c(t)$  and  $|c(t) - c_+| \leq C\epsilon e^{-bt}$ . Also, there exists

$$\gamma_+ = \lim_{t \rightarrow \infty} (\gamma(t) - \int_0^t (c(s) - c_+) ds).\tag{6.8}$$

We put

$$\tilde{\gamma}(t) = - \int_0^t (c(s) - c_+) ds + \gamma(t) - \gamma_+.\tag{6.9}$$

Then, it follows from (5.3)

$$\begin{aligned}&\|u(\cdot + c_+ t - \gamma_+, t) - u_{c_+}(\cdot)\|_{H_a^1} \\ &\leq \|u(\cdot + c_+ t - \gamma_+, t) - u_{c(t)}(\cdot + \tilde{\gamma}(t))\|_{H_a^1} + \|u_{c(t)}(\cdot + \tilde{\gamma}(t)) - u_{c_+}(\cdot)\|_{H_a^1} \\ &\leq \|v(\cdot + \tilde{\gamma}(t), t)\|_{H_a^1} + C(|c(t) - c_+| + |\tilde{\gamma}(t)|) \\ &\leq C\|w(\cdot, t)\|_{H^1} + C(|c(t) - c_+| + |\tilde{\gamma}(t)|) \\ &\leq C\epsilon e^{-bt}.\end{aligned}\tag{6.10}$$

This completes the proof  $\square$

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